2023

MATHEMATICS

Paper : MATHC3076

(PDE and Systems of ODE)

Full Marks : 60

Pass Marks : 24

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer (any *five*) : 1×5=5

(a) The general form of a first-order quasi-linear partial differential equation is

(i) $a(x, y)u_x + b(x, y)u_y = c(x, y, u)$

(ii) $a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$

(iii) $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$

(iv) $a(x, y)u_x^2 + b(x, y)u_y^2 = c(x, y, u)$

(b) A partial differential equation can be formed by

(i) eliminating arbitrary constant only

(ii) eliminating arbitrary constant and arbitrary function *(iii)* eliminating arbitrary function only

(iv) integrating the given family of curves

(c) Which of the following represents canonical form of a parabolic equation?

(i) $u_{\eta\eta} = 0$

(ii) $u_{\xi\eta} = 0$

(iii) $u_{\alpha\alpha} + u_{\beta\beta} = 0$

(iv) $u_{\xi\eta} + u_{\eta\eta} = 0$

(d) The equation $u_t - c^2(u_{xx} + u_{yy}) = 0$, where $c \neq 0$, is known as

(i) wave equation

(ii) Laplace's equation

(iii) heat equation

(iv) Poisson's equation

(e) If the system of two linear differential equations in two unknown functions

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + F_1(t)$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + F_2(t)$$

is to be homogeneous, then

(i) $F_1(t) < F_2(t), \forall t$

(ii) $F_1(t) = F_2(t) = k, k \neq 0, \forall t$

(iii) $F_1(t) > F_2(t), \forall t$

(iv) $F_1(t) = F_2(t) = 0, \forall t$

(f) Which of the following is a non-homogeneous wave equation?

(i) $u_{tt} = c^2 u_{xx}, c \neq 0$

(ii) $u_{tt} = c^2 u_{xx} + h(x,t), c \neq 0$

(iii) $u_{tt} + u_{xx} = 0$

(iv) $u_t = c^2 u_{xx}, c \neq 0$

(g) Which of the following does not come under partial differential equations?

(i) Laplace's equation

(ii) One-dimensional wave equation

(iii) Heat equation

(iv) Equations of motion

(h) Two solutions

$$\begin{matrix} x = f_1(t) \\ y = g_1(t) \end{matrix} \quad \text{and} \quad \begin{matrix} x = f_2(t) \\ y = g_2(t) \end{matrix}$$

of a homogeneous linear system of two differential equations in two unknown functions x and y are linearly independent on an interval $[a, b]$ if

(i) $f_1(t)f_2(t) - g_1(t)g_2(t) \neq 0, \forall t \in [a, b]$

(ii) $f_1(t)g_1(t) - f_2(t)g_2(t) \neq 0, \ \forall \ t \in [a, b]$

(iii) $f_1(t)g_2(t) - f_2(t)g_1(t) \neq 0, \ \forall \ t \in [a, b]$

(iv) $f_1(t)g_2(t) + f_2(t)g_1(t) = 0, \ \forall \ t \in [a, b]$

(i) Which of the following is correct?

(i) A hyperbolic equation has one real family of characteristics.

(ii) For an elliptic equation there does not exist any real characteristics.

(iii) For a parabolic equation there does not exist any real characteristics.

(iv) An elliptic equation has two real and distinct families of charactersitics.

(j) The order of the partial differential equation

$$\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial y^3}\right\}^{\frac{1}{2}} = 2x\left(\frac{\partial^2 u}{\partial x^2}\right)^2$$

is

(i) 3

(ii) 1

(iii) 4

(iv) 2

2. Answer the following questions (any *five*) : 2×5=10

(a) Find the partial differential equation arising from the surface

$$z = xy + f(x^2 + y^2)$$

(b) Give geometrical interpretation of first-order linear partial differential equation.

(c) If D is a differential operator with respect to t, then find $(D^2+1)(3D+2)t^3$.

(d) Show that the family of spheres $x^2 + y^2 + (z-c)^2 = \pi^2$ satisfies the first-order linear partial differential equation $yp - xq = 0$.

(e) Determine the regions in which the equation $u_{xx} + y^2 u_{yy} = y$ is parabolic and elliptic.

(f) Write the normal form of a linear system of n differential equations in n unknown functions $x_1, x_2, x_3, \ldots, x_n$.

(g) Transform the single linear differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = 0$$

into a system of first-order differential equations.

3. Answer the following questions (any *five*) : 5×5=25

(a) Find the general solution of the linear equation $x^2u_x + y^2u_y = (x+y)u$.

(b) Applying the method of separation of variables, solve the following equation :

$$u_x + 2u_y = 0, \quad u(0, y) = 3e^{-2y}$$

(c) Solve the following system of equations by using operator method :

$$\frac{dx}{dt} + \frac{dy}{dt} - x - 3y = e^t$$

$$\frac{dx}{dt} + \frac{dy}{dt} + x = e^{3t}$$

(d) Show that $u_1 = e^x$ and $u_2 = e^{-y}$ are solutions of the non-linear equation $(u_x + u_y)^2 - u^2 = 0$ but their sum $(e^x + e^{-y})$ is not a solution of this equation.

(e) Derive one-dimensional heat equation in the form

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(f) Determine the integral surface of the equation

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the data $x + y = 0$, $u = 1$.

(Continued)

(g) Find the general solution of the following system of equations :

$$\frac{dx}{dt} = 3x + y$$

$$\frac{dy}{dt} = 4x + 3y$$

(h) Find the first three approximations of the solution by using the method of successive approximations in the following initial-value problem

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0$$

(i) Find the solution of the initial-value systems

$$u_t + 3uu_x = v - x, \quad v_t - cv_x = 0$$

with $u(x, 0) = x$ and $v(x, 0) = x$.

4. Answer the following questions (any *two*) : 10×2=20

(a) Find the characteristic equations and characteristic curves, and then reduce the equation

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$$

to canonical form. 2+1+7=10

(Turn Over)

(b) Find the solution of

$$u_{xx} - u_{yy} = 1$$
$$u(x, 0) = \sin x$$
$$u_y(x, 0) = x$$

(c) Applying the modified Euler method to

$$\frac{dy}{dx} = x^2 + y^2$$
$$y(0) = 1$$

approximate the values of the solution y at $x = 0 \cdot 1$ and $0 \cdot 2$, using $h = 0 \cdot 1$. Obtain the results to three figures after the decimal point.

(d) Solve the initial boundary-value problem

$$u_t = ku_{xx}, \quad 0 < x < l, \quad t > 0$$
$$u(0, t) = 0, \quad t \geq 0$$
$$u(l, t) = 0, \quad t \geq 0$$
$$u(x, 0) = x(l - x), \quad 0 \leq x \leq l$$

by using the method of separation of variables.

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